BOURGIN-YANG VERSIONS OF THE BORSUK-ULAM THEOREM FOR p-TORAL GROUPS.

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ABSTRACT. Let $G = \mathbb{Z}_p^k$ be the p-torus of rank k, p a prime, or respectively $G = \mathbb{T}^k = (S^1)^k$ be a k-dimensional torus and let V and W be orthogonal representations of G with $V^G = W^G = \{0\}$. Let $S(V)$ be the sphere of V and suppose $f : S(V) \to W$ is a G-equivariant mapping. We give an estimate for the dimension of the set $Z_f = f^{-1}\{0\}$ in terms of $\dim V$ and $\dim W$. This extends the Bourgin-Yang version of the Borsuk-Ulam theorem onto this class of groups. Also we provide a sufficient and necessary condition for the existence of equivariant map between spheres of two orthogonal representations of such a group. Finally, we show that for any p-toral group G and a G-map $f : S(V) \to W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$.

1. INTRODUCTION

Let G be a compact Lie group and let V, W be two orthogonal representations of G such that $V^G = W^G = \{0\}$ for the sets of fixed points of G. Let $f : S(V) \to W$ be a G-equivariant mapping. By Z_f, we denote the set

$$Z_f := \{v \in S(V) \mid f(v) = 0\}.$$

The problem of estimating the covering dimension of the set Z_f was considered firstly by C. T. Yang [19, 20] and (independently) D. G. Bourgin [5] for the case $G = \mathbb{Z}_2$. Specifically they proved that for a $\mathbb{Z}_2$-equivariant mapping f from the unit sphere $S(\mathbb{R}^n)$ in $\mathbb{R}^n$ into $\mathbb{R}^m$, where the Euclidean spaces are considered as representations of $\mathbb{Z}_2$ with the antipodal action,

$$\dim Z_f \geq n - m - 1,$$

where dim is the covering dimension. Consequently, it generalized the classical Borsuk-Ulam theorem. In [9], Dold extended the Bourgin-Yang problem to a fibre-wise setting, giving an estimate for the set $Z_f = f^{-1}\{0\}$, where $\pi : E \to B$ and $\pi' : E' \to B$ are vector bundles and $f : S(E) \subset E \to E'$ is a $\mathbb{Z}_2$-map, which preserve fibres ($\pi' \circ f = \pi$). In [11] and [15] this problem was considered for the case of the cyclic group $G = \mathbb{Z}_p$ (p prime), and in [14] for bundles $E \to B$ whose fibre has the same cohomology (mod p) of a product of spheres.

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In all these cases, if B is a single point, Bourgin-Yang versions of the Borsuk-Ulam theorem are obtained for $G = \mathbb{Z}_p$, with p prime.

Recently, in [13] the authors considered the Bourgin-Yang problem for the case that G is a cyclic group of a prime power order, $G = \mathbb{Z}_{p^k}$, $k \geq 1$. Using the result of [1] they proved the following theorem [13, Theorem 1.1]:
Let V, W be two orthogonal representations of the cyclic group $\mathbb{Z}_{p^k}$ and $f : S(V) \to W$ an equivariant map. Then, the covering dimension $\dim(Z_f) = \dim(Z_f/G) \geq \phi(V, W)$, where ϕ is a function depending on $\dim V$, $\dim W$ and the orders of the orbits of actions on $S(V)$ and $S(W)$ (cf. [13, Theorems 3.6 and 3.9]). In particular, if $\dim W < \dim V/p^{k-1}$, then $\phi(V, W) \geq 0$, which means that there is no G-equivariant map from $S(V)$ into $S(W)$.

In this paper we study the Bourgin-Yang problem for groups $G = \mathbb{Z}_p^k$ the p-torus of rank k, p a prime, and $G = \mathbb{T}^k = (S^1)^k$ the k-dimensional torus respectively. Also, we provide a sufficient and necessary condition for the existence of an equivariant map between spheres of two orthogonal representations of such groups. Finally, we show that for any p-toral group G and a G-map $f : S(V) \to W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$.

The plan of the paper is as follows. In Section 2, we consider the case $G = \mathbb{Z}_p^k$, the p-torus of rank k, p a prime. In Theorem 2.3, we give an estimate of the covering dimension of the set

$$\begin{aligned} Z_f^{H_\alpha} &:= \{v \in S(V^{H_\alpha}) \,|\, f^{H_\alpha}(v) = 0\,, \text{ with } f^{H_\alpha} = f_{|S(V^{H_\alpha})} : S(V^{H_\alpha}) \to W^{H_\alpha}\} \\ &= \quad Z_f \cap S(V^{H_\alpha}), \end{aligned}$$

for every maximal isotropy group $H_\alpha \subset G$ of the action on $S(V)$. In Theorem 2.5, we provide a necessary and sufficient condition for the existence of a $\mathbb{Z}_p^k$ -equivariant map $f : S(V) \to S(W)$. Theorem 2.6 states that for any $\mathbb{Z}_p^k$-map $f : S(V) \to W$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$. Using the length index of a G-space with respect to an equivariant cohomology theory (cf. [3]) and the Borel localization theorem we finalize this section proving Theorem 2.7, which is the correspondent Bourgin-Yang theorem for the p-torus. In Section 3, considering $G = \mathbb{T}^k = (S^1)^k$, the k-dimensional torus, the first main result is Theorem 3.7, describing an obstruction for the existence of S^1-equivariant map between the spheres of two orthogonal representations of S^1. As a consequence of Theorem 3.7, we obtain Theorem 3.10, which is the version of the Bourgin-Yang theorem for the torus and which also provides a necessary and sufficient condition for the existence of a $\mathbb{T}^k$- equivariant map between spheres of orthogonal representations of the torus. Theorems 3.12 and 3.13 state that for any $(S^1)^k$-map $f : S(V) \to W$, $k \geq 1$, with $\dim V = \infty$ and $\dim W < \infty$, we have $\dim Z_f = \infty$. In Section 4, we have Theorem 4.2 [Characterization of p-toral groups] as main result, which shows that Theorems 2.6 and 3.13 can be extended on a larger class of groups called p-toral.

Next, we recall some results and fix some notations. For $G = \mathbb{Z}_p^k$, with p odd, and $G = \mathbb{T}^k$ every nontrivial irreducible orthogonal representation is even dimensional and admits the complex structure ([18]), thus V and W admit it too. Put $d(V) = \dim_{\mathbb{C}} V = \frac{1}{2}\dim_{\mathbb{R}} V$, and correspondingly $d(W) = \dim_{\mathbb{C}} W = \frac{1}{2}\dim_{\mathbb{R}} W$, are integral numerical invariants of V, and respectively of W. If $G = \mathbb{Z}_2^k$ and V,W are orthogonal representations of G, then we put $d(V) = \dim_{\mathbb{R}} V$, and respectively $d(W) = \dim_{\mathbb{R}} W$. For a G space X with $X^G = \emptyset$, let $H = G_x$, $x \in X$ be a maximal isotropy subgroup in X. Since G is abelian, the set $X^H = X^{(H)}$

is an G-invariant closed subset of X. Moreover, the action of G on X^H factorizes through the induced action of $K := G/H$.

2. Case of G being a p-torus

Now, let us take $X = S(V)$, where V is an orthogonal representation of $G = \mathbb{Z}_2^k$, or $G = \mathbb{Z}_p^k$, with odd p, such that $V^G = \{0\}$. In the later case, V admits a complex structure. Let

$$V = \bigoplus_{\alpha} l_\alpha V_\alpha = \bigoplus_{\alpha} V^\alpha, \quad V^\alpha := l_\alpha V_\alpha \tag{1}$$

be the decomposition of V into multiplies of irreducible representations of G. Note that every irreducible representation V_α of $G = \mathbb{Z}_2^k$ is given by a homomorphism $\xi_\alpha : G \to \mathbb{Z}_2 = O(1) = S^0 = S(\mathbb{R})$, and correspondingly, every irreducible representation of $G = \mathbb{Z}_p^k$ is given by a homomorphism $\xi_\alpha : G \to \mathbb{Z}_p \subset S^1 = S(\mathbb{C})$. Observe that $H_\alpha = \ker \xi_\alpha$ is a group of rank $k-1$ in G, i.e. it is isomorphic to $\mathbb{Z}_2^{k-1}$, or $\mathbb{Z}_p^{k-1}$, respectively. In other words, each H_α is a maximal subgroup of G.

Furthermore, we have $V^\alpha \subset V^{H_\alpha}$, and for a maximal isotropy group $H = H_\alpha$,

$$V^H = \bigoplus_{H_\beta = H_\alpha} V^\beta,$$

where the summation is over all representations appearing in V, with the kernel equal to H_α. Since all H_α are maximal subgroups, not only as the isotropy subgroups, but as subgroups of G different from G, (i.e., of co-rank 1), the linear subspaces

$$V^{H_{\alpha_1}} \cap V^{H_{\alpha_2}} = \{0\}\,,$$

because $V^{H_{\alpha_1}} \cap V^{H_{\alpha_2}} = V^{(H_{\alpha_1}, H_{\alpha_2})}$, where $(H_{\alpha_1}, H_{\alpha_2})$ is the subgroup generated by H_{α_1} and H_{α_2}. But, the latter is equal to G, thus $V^{(H_{\alpha_1}, H_{\alpha_2})} = V^G = \{0\}$, and these two spaces are orthogonal, since the action is orthogonal.

Grouping all such irreducible representations in one term, we can rewrite (1) as

$$V = \bigoplus_{\alpha} l_\alpha V_\alpha = \bigoplus_{\alpha} V^\alpha = \bigoplus_{H} V^H\,, \tag{2}$$

where the summation is taken over all maximal isotropy groups H, thus $H = H_\alpha$, for some ξ_α. Consequently, $\dim_{\mathbb{R}} V^H = \sum_{H_\beta = H} l_\beta$, if $G = \mathbb{Z}_2^k$, and $\dim_{\mathbb{R}} V^H = 2 \sum_{H_\beta = H} l_\beta$, if $G = \mathbb{Z}_p^k$.

Let us denote $\sum_{H_\beta = H} l_\beta$ by $l_H(V)$ and we summarize the above results in the following well-known facts.

Fact 2.1. *For every orthogonal representation of $G = \mathbb{Z}_p^k$, p a prime, with $V^G = \{0\}$, we have the canonical decomposition into irreducible representations (each of them possessing a complex structure consistent with the action, if $p > 2$):*

$$V = \bigoplus_{\alpha} l_\alpha V_\alpha = \bigoplus_{\alpha} V^\alpha = \bigoplus_{\alpha} V^{H_\alpha}\,,$$

where at the latter H_α ranks all maximal isotropy subgroups of $S(V)$ and $V^{H_\alpha} = \bigoplus_{\beta : H_\beta = H_\alpha} V^\beta$.

Consequently,

$$d(V) = \sum_{\alpha} l_\alpha(V) = \sum_{H_\alpha} l_{H_\alpha}(V) = \frac{1}{2}\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V, \tag{3}$$

if p is a odd prime, or

$$d(V) = \sum_{\alpha} l_\alpha(V) = \sum_{H_\alpha} l_{H_\alpha}(V) = \dim_{\mathbb{R}} V \tag{4}$$

if $p = 2$, respectively and the spheres $S(V^\alpha) = S(V^{H_\alpha})$ form a family of disjoint invariants subsets of $S(V)$.

Analogous to the torus case, the homomorphisms $\xi_\alpha : \mathbb{Z}_p^k \to \mathbb{Z}_p \subset U(1) = S^1$, correspondingly to $\xi\alpha : \mathbb{Z}_2^k \to \mathbb{Z}_2 \subset O(1) = S^0$, is called *a weight of* G. The number l_α is called the multiplicity of weight ξ_α.

Remark 2.2. For $G = \mathbb{Z}_p$, all nontrivial irreducible representations $\{V_j\}_{j=1}^{p-1}$ are of the form

$$V_j \simeq \mathbb{C} \,:\, g \mapsto \exp(2\pi\,\imath\, j/p) \in S^1 \subset \mathbb{C}\,.$$

Consequently, for every $1 \le j \le p-1$, the homomorphism ξ_j is a monomorphism, thus $H_j = \mathbf{e}$, for every j, which is obviously the unique maximal subgroup of G.

Denote by $Z_f^{H_\alpha}$ the set

$$\begin{aligned} Z_f^{H_\alpha} &:= \{v \in S(V^{H_\alpha}) \,|\, f^{H_\alpha}(v) = 0\,,\ \text{with } f^{H_\alpha} = f_{|S(V^{H_\alpha})} : S(V^{H_\alpha}) \to W^{H_\alpha}\} \\ &= \quad Z_f \cap S(V^{H_\alpha})\,. \end{aligned}$$

We have the following.

Theorem 2.3. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$, with $V^G = W^G = \{0\}$, and let $f : S(V) \to W$ be an equivariant map. Then, for every maximal isotropy group $H_\alpha \subset G$ of the action on $S(V)$, we have $f(S(V^{H_\alpha})) \subset W^{H_\alpha}$, and for the covering dimension*

$$\dim(Z_f^{H_\alpha}) = \dim(Z_f^{H_\alpha}/G) \ge 2[d(V^{H_\alpha}) - d(W^{H_\alpha})] - 1 = 2[l_{H_\alpha}(V) - l_{H_\alpha}(W)] - 1\,,$$

if p is an odd prime, and

$$\dim(Z_f^{H_\alpha}) = \dim(Z_f^{H_\alpha}/G) \ge d(V^{H_\alpha}) - d(W^{H_\alpha}) - 1 = l_{H_\alpha}(V) - l_{H_\alpha}(W) - 1\,,$$

if $p = 2$, respectively.

Proof. Let H be a maximal isotropy subgroup in $S(V)$ and $f^H : S(V^H) \to W^H$ the restriction of f to $S(V^H)$. Note that V^H is a sub-representation of G, the action of G on $S(V^H)$ factorizes through $K = G/H$. Here $H \simeq \mathbb{Z}_p^{k-1}$, or $\simeq \mathbb{Z}_2^{k-1}$. Consequently, $G/H \simeq \mathbb{Z}_p$, or $\mathbb{Z}_2$, respectively. Moreover, $f : S(V^H) \to W^H$ is a K-equivariant map. Now, from the main result of [13] applied to the cyclic group $\mathbb{Z}_p$ of prime order, it follows that

$$\dim(Z_f^{H_\alpha}) = \dim(Z_f^{H_\alpha}/G) \ge 2[d(V^{H_\alpha}) - d(W^{H_\alpha})] - 1,$$

if p is an odd prime, or

$$\dim(Z_f^{H_\alpha}) = \dim(Z_f^{H_\alpha}/G) \geq d(V^{H_\alpha}) - d(W^{H_\alpha}) - 1,$$

if $p = 2$, respectively. □

Remark 2.4. By Fact 2.1, we have $\sum\limits_{H_\alpha} \dim_{\mathbb{C}} V^{H_\alpha} = \sum\limits_{H_\alpha} d(V^{H_\alpha}) = \dim_{\mathbb{C}} V = d(V)$ for p odd, and $\sum\limits_{H_\alpha} \dim_{\mathbb{R}} V^{H_\alpha} = \sum\limits_{H_\alpha} d(V^{H_\alpha}) = \dim_{\mathbb{R}} V = d(V)$ if $p = 2$ respectively. Summing up the inequalities of Theorem 2.3 we get

$$\sum_\alpha \dim Z_f^{H_\alpha} \geq 2(\,d(V) - \sum_\alpha d(W^{H_\alpha})\,) - c \geq 2(d(V) - d(W)) - c,$$

or

$$\sum_\alpha \dim Z_f^{H_\alpha} \geq\, d(V) - \sum_\alpha d(W^{H_\alpha}) \,- c \geq d(V) - d(W) - c$$

depending whether p is odd prime, or $p = 2$, where $c = \sum\limits_\alpha 1$.

A better estimate to $\sum\limits_\alpha \dim Z_f^{H_\alpha}$ will be given in Theorem 2.7.

Theorem 2.5. *A necessary and sufficient condition for the existence of a $\mathbb{Z}_p^k$-equivariant map $f : S(V) \to S(W)$ is*

$$\dim V^{H_\alpha} \leq \dim W^{H_\alpha}$$

for every maximal isotropy subgroup H_α on $S(V)$. In particular, if $d(W) < d(V)$, then there is no G-equivariant map from $S(V)$ into $S(W)$.

Proof. Observe that if there is a G-map $f : S(V) \to S(W)$, then $Z_f = \emptyset$, which gives $\dim Z_f^{H_\alpha} = -1$, for every H_α. By Theorem 2.3, we get $l_{H_\alpha}(V) \leq l_{H_\alpha}(W)$, for every maximal isotropy group H_α, which appears in the decomposition of V.

If $d(V) = \sum l_{H_\alpha}(V) > \sum l_{H_\alpha}(W) = d(W)$, then there exists at least one index α_0, for which $l_{H_{\alpha_0}}(V) > l_{H_{\alpha_0}}(W)$. By Theorem 2.3, for a G-map $f : S(V) \to S(W) \subset W$ it implies that $\dim Z_f^{H_{\alpha_0}} \geq 0$ and, consequently, $Z_f \neq \emptyset$, which gives a contradiction.

The converse was already proved in [12, Theorem 2.5], in another formulation. We will present a proof of it. It is enough to show that for every maximal subgroup $H \subset G$, under the assumption $0 < d(V^H) \leq d(W^H)$, there exists $\mathbb{Z}_p$-equivariant map $f^H : S(V^H) \to S(W^H)$. Indeed, once more, using the fact that the action of G on $S(V^H)$ and $S(W^H)$ factorizes through $K = G/K \simeq \mathbb{Z}_p$, and any such a map f^H is G-equivariant. Now, it is sufficient to take the joint of maps of the corresponding joints of

$$\mathop{*}_H f^H : S(V) = S(\bigoplus_H V^H) = \mathop{*}_H S(V^H) \;\to\; \mathop{*}_H S(W^H) = S(\bigoplus_H W^H) = S(W)\,.$$

By Remark 2.2, $V^H = \bigoplus\limits_{j=1}^{p-1} l_j\, V_j$, where $l_j \geq 0$, and $W^H = \bigoplus\limits_{j=1}^{p-1} \tilde{l}_j\, V_j$,where $\tilde{l}_j \geq 0$. Since $d(V^H) = \sum\limits_{j=1}^{p-1} l_j \leq \sum\limits_{j=1}^{p-1} \tilde{l}_j = d(W^H)$, it is enough to show that for every $1 \leq j_1, j_2 \leq p-1$,

there exist a $\mathbb{Z}_p$-equivariant map from $S(V_{j_1}) \to S(V_{j_2})$. Let $1 \le j_1^{-1} \le p-1$ be the inverse of j_1 in $\mathbb{Z}_p^*$. It is easy to check that the map

$$S^1 \ni z \mapsto z^{j_1^{-1} j_2} \in S^1$$

is the required $\mathbb{Z}_p$-equivariant map. $\square$

Let $H_1, H_2, \ldots, H_s$ be all maximal isotropy subgroups on $S(V)$ and $H_{i_1}, H_{i_2}, \ldots, H_{i_r}$ be all among $H_1, \ldots, H_s$, for which $l_{H_{i_j}}(W) > 0$. Theorem 2.3 leads to the following.

$$\dim Z_f \;\ge\; \max_{i=1,\ldots,s} \{2(l_{H_i}(V) - l_{H_i}(W)) - 1\} =$$

$$= \max\left(\max_{j=1,\ldots,r} \{2(l_{H_{i_j}}(V) - l_{H_{i_j}}(W)) - 1\},\ \max_{i \notin \{j_1,\ldots,j_r\}} \{2\, l_{H_i}(V) - 1\} \right). \tag{5}$$

Here $l_{H_i}(V)$ denotes the multiplicity in V of the irreducible representation corresponding to H_i, and is equal to $\dim_{\mathbb{C}} V^{H_i}$, i.e. in the first equality $l_{H_i}(W) = 0$, if $W^{H_i} = \{0\}$.

Note that the estimate (5) is weaker then expected $\dim Z_f \ge 2(d(V) - d(W)) - 1$ which holds for $G = \mathbb{Z}_p$, or $\dim Z_f \ge d(V) - d(W) - 1$ which holds for $G = \mathbb{Z}_2$ respectively. We shall show that this stronger version also holds for $\mathbb{Z}_p^k$, $k \ge 2$ (cf. Theorem 2.7).

Now, we turn to the problem of existence of G-equivariant maps from the sphere $S(V)$ of infinite dimensional representation V, into the sphere $S(W)$ of a finite dimensional representation W, or the estimate of dimension of the set Z_f, for an equivariant map $f : S(V) \to W$.

Theorem 2.6. *Let V, W be an orthogonal representations of a p-torus $G = \mathbb{Z}_p^k$, p prime, such that $V^G = \{0\} = W^G$. If $\dim V = \infty$ and $\dim W < \infty$, then for every G-equivariant map $f : S(V) \to W$ we have*

$$\dim Z_f = \infty .$$

In particular, there is no G-equivariant map $S(V) \to S(W)$ under this assumption.

Proof. For a given $d \in \mathbb{N}$, let us take sub-representation $V(d) \subset V$ such that $d(V(d)) \ge d$. We have only a finite number of subgroups $H \subset G$ and

$$d(V) = \sum_H l_H(V) = \sum_H \dim V^H ,$$

where the sum is taken over all maximal isotropy subgroups, there exists $H_0 \subset G$ such that $d(V(d)^{H_0}) = \dim_{\mathbb{C}} V(d)^{H_0} \to \infty$, if $d(V) \to \infty$.

Now, using estimate (5) we get

$$\dim Z_f \ge \lim_{d\to\infty} 2(d(V(d)^{H_0}) - d(W^{H_0})) - 1 = \infty ,$$

or

$$\dim Z_f \ge \lim_{d\to\infty} d(V(d)^{H_0}) - d(W^{H_0}) - 1 = \infty,$$

if $p = 2$, which proves the statement. $\square$

We have the following strengthened version of Theorem 2.3 which is the version of the Bourgin-Yang theorem for p-torus.

Theorem 2.7. *Let V, W be two orthogonal representations of the group $G = \mathbb{Z}_p^k$ such that $V^G = W^G = \{0\}$. Let $f : S(V) \to W$ be a G-equivariant map and $\{H_\alpha\}$, $1 \leq \alpha \leq s$, all maximal isotropy subgroups of G in $S(V)$. Then, either*

$$\sum_\alpha \dim Z_f^{H_\alpha} \geq 2(\, d(V) - d(W)\,) - 1\,,$$

or

$$\sum_\alpha \dim Z_f^{H_\alpha} \geq \, d(V) - d(W) \, - 1$$

depending whether p is odd prime, or $p = 2$.

A proof follows the scheme of the proof of the corresponding Bourgin-Yang theorem for $G = \mathbb{Z}_{p^k}$ given in [13]. It requires a notion of the length index of a G-space with respect to an equivariant cohomology theory (cf. [3]).

We briefly recall the notion of an equivariant index based on the cohomology length in a given cohomology theory. It was introduced and described in details by Thomas Bartsch in [3, Chapter 4]. He presented a very general version of the mentioned index, considering an equivariant map between two pairs of G-spaces, and defining an index for this triple. We consider the case when these two pairs are equal and the map is equal to the identity. Moreover, we study this notion taking as an equivariant cohomology theory the equivariant Borel cohomology of the Spanier-Alexander cohomology, denoted by $H_G^*(X)$ (see [3], or [6] for more information).

Let (X, X') be a pair of G-spaces. The Borel-Alexander-Spanier cohomology of (X, X') in a coefficient ring $\mathcal{R}$, denoted as $H_G^*(X, X'; \mathcal{R})$, are defined as

$$H_G^*(X, X'; \mathcal{R}) = H^*(X \times_G EG,\, X' \times_G EG; \mathcal{R}),$$

where on the right hand side are Alexander-Spanier cohomology with coeeficients in $\mathcal{R}$. If $\mathcal{R}$ is fixed, then we omit it in the notation, writing shortly $H_G^*(X, X')$.

Let us fix a set $\mathcal{A}$ of G-spaces. Usually, it is a family of orbits, which obviously is finite, if G is finite.

Recall that for the equivariant cohomology theory H_G^* and a G-pair (X, X'), the cohomology $H_G^*(X, X')$ is a module over the coefficient ring $H_G^*(\mathrm{pt}) = H^*(BG; \mathcal{R})$, via the natural G-map $p_X : X \to \mathrm{pt}$. We write

$$\omega \cdot \gamma = p_X^*(\omega) \cup \gamma, \text{ and } \omega_1 \cdot \omega_2 = \omega_1 \cup \omega_2,$$

for $\gamma \in H_G^*(X, X')$ and ω_1, $\omega_2 \in H_G^*(\mathrm{pt})$.

For a prime number p, we denote by F_p the ring (actually a field) of integers modulo p to distinguish it from the cyclic group $\mathbb{Z}_p$, considered as a group only. For the group $G = \mathbb{Z}_p^k$, p odd prime, and the ring $\mathcal{R} = F_p$, we have the following classical description of $H_G^*(\mathrm{pt}; F_p)$:

$$H^*(BG; F_p) = F_p[c_1, c_2, \ldots, c_k] \otimes_{F_p} E(w_1, w_2, \ldots, w_k),$$

where $F_p[c_1, c_2, \ldots, c_k]$ is the polynomial algebra with generators $c_i \in H^2(BG; F_p)$ and $E(w_1, w_2, \ldots, w_k)$ is the exterior F_p-algebra of k variables $w_1, w_2, \ldots, w_k$, $w_i \in H^1(BG; F_p)$, $1 \leq i \leq k$ (see [3] for more references).

For $p = 2$, $G = \mathbb{Z}_2^k$ and $\mathcal{R} = F_2$, the corresponding description of $H_G^*(\mathrm{pt}; F_2)$ is even simpler:

$$H^*(BG; F_2) = F_2[w_1, w_2, \ldots, w_k],$$

where for $1 \le i \le k$, $w_i \in H^1(BG; F_p)$ is the image of $w \in H^1(B\mathbb{Z}_2; F_p)$, the generator of $H^*(B\mathbb{Z}_2; F_p)$ by the homomorphism $p_i^* : H^*(B\mathbb{Z}_2; F_p) \to H^*(B\mathbb{Z}_2^k; F_p)$.

For a definition of the length index, we have to fix an ideal I in the ring $R = H_G^*(\mathrm{pt}; F_p)$ and a family $\mathcal{A}$ of G-sets.

First, we fix the family of G-sets (actually a family of orbits) taking

$$\mathcal{A} = \{G/H : \, H \subsetneq G \;\text{ a subgroup}\}.$$

Next, taking for $G = \mathbb{Z}_p^k$, p odd, the ring $R := H_G^*(\mathrm{pt})$, and $I = (c_1, c_2, \ldots, c_k)$, the ideal generated by $c_1, c_2, \ldots, c_k$, or correspondingly, for $G = \mathbb{Z}_2^k$, the ring $R = F_p[w_1, w_2, \ldots, w_k]$, and the ideal $I = R = F_p[w_1, w_2, \ldots, w_k]$, we obtain the following adjustment of [3, Definition 4.1].

Definition 2.8. *The $(\mathcal{A}, H_G^*, I)$ – length index of a pair (X, X') of G-spaces is the smallest r such that there exist $A_1, A_2, \ldots, A_r \in \mathcal{A}$ with the following property:*
For all $\gamma \in H_G^*(X, X')$ and all $\omega_i \in I \cap \ker(H_G^*(\mathrm{pt}) \to H_G^*(A_i))$, $i = 1, 2, \ldots, r$, the product

$$\omega_1 \cdot \omega_2 \cdot \cdots \cdot \omega_r \cdot \gamma = 0 \in H_G^*(X, X').$$

The $(\mathcal{A}, H_G^*, I)$ – *length index* has many properties, which are important from the point of view of applications to study critical points of G-invariant functions and functionals (see [3]). Next, we shall use only a part of them.

Remark 2.9. *By Observation 5.5 of* [3] *we can replace $\mathcal{A}$ by a smaller family $\mathcal{A}' \subset \mathcal{A}$, provide for every $A \in \mathcal{A}$, there exist $A' \in \mathcal{A}$ and a G-map $\eta : A \to A'$. In particular, for $G = \mathbb{Z}_p^k$, instead of the family $\mathcal{A} = \{H \subsetneq G\}$ we can take the family $\mathcal{A}' = \{H \subsetneq G, H \text{ maximal}\}$.*

We begin with the following.

Proposition 2.10 ([3, Theorem 5.2])**.** *Consider the group $G = \mathbb{Z}_2^k$, $k \ge 1$, and let l denote $\{\mathcal{A}, H_G^*, I\}$ - length index of Definition 2.8. Let V be an orthogonal representation of G such that $V^G = \{0\}$. Then*

$$l(S(V)) = d(V) = \dim_{\mathbb{R}} V\,.$$

Proposition 2.11 ([3, Theorem 5.3])**.** *Consider the group $G = \mathbb{Z}_p^k$, $k \ge 1$, p an odd prime, and let l denote $\{(\mathcal{A}, H_G^*, I\}$-length index of Definition 2.8. Let V be an orthogonal representation of G such that $V^G = \{0\}$. Then,*

$$l(S(V)) = d(V) = \frac{1}{2}\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V\,.$$

To prove Theorem 2.7, we need to discuss a relation between the length index $l = (\mathcal{A}, H_G^*, I)$ of the G-set X and its dimension.

Let X is a compact G-space and G a compact Lie group. In [17, Proposition 5.3, page 147], G. Segal discussed the equivariant K-theory. To do it, he showed that there is the Atiyah-Hirzebruch spectral sequence for equivariant K-theory,

$$E_2^{s,t} = H^s(X/G; \mathcal{K}_G^t) \Rightarrow K_G^*(X)\,.$$

For a proof of this convergence he used an invariant filtration of X such that $K_G^*(X)$ is the associated module of the limit of this spectral sequence, with respect to this filtration. In [13], we used this filtration to describe properties of a length index in K_G^*-theory and prove

the Bourgin-Yang theorem for $G = \mathbb{Z}_{p^k}$. Here, we also use the same filtration to establish some properties of the length index $l = (\mathcal{A}, H^*_G, I)$.

If X is a G-CW-complex, which is filtered by its skeletons $\{X^s\}$, it is customary to define a filtration of $H^*_G(X)$ by setting $H^*_{G,s}(X) = \ker(H^*_G(X) \to H^*_G(X^{s-1}))$. It corresponds to a filtering of X by the G-subspaces $\pi^{-1}(Y^s)$, when the orbit space $Y = X/G$ is a CW-complex and $\{Y^s\}$ its skeletons ($\pi : X \to Y$ is the projection).

The general case is discussed in [17, §5], by use of the nerve of a G-stable closed finite covering of X. For each finite covering $\mathcal{U} = \{U_j\}_{j\in\mathcal{S}}$ of a compact G-space X by G-stable closed sets it is associated a compact G-space $\mathcal{W}_\mathcal{U}$, with a G-map $w : \mathcal{W}_\mathcal{U} \to X$ and a filtration by G-subspaces $\mathcal{W}^0_\mathcal{U} \subset \mathcal{W}^1_\mathcal{U} \subset \cdots \subset \mathcal{W}^j_\mathcal{U} \subset \cdots \subset \mathcal{W}_\mathcal{U}$, so that the following conditions are satisfied:

(i) $w^* : H^*_G(X) \to H^*_G(\mathcal{W}_\mathcal{U})$ is an isomorphism, and
(ii) when $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a G-map $\mathcal{W}_\mathcal{U} \to \mathcal{W}_\mathcal{V}$, defined up to G-homotopy, respecting the filtrations and the projections onto X.

Definition 2.12. *We say that an element of $H^*_G(X)$ is in $H^*_{G,s}(X)$ if, for some finite covering $\mathcal{U}$, it is in the kernel of $w^* : H^*_G(X) \to H^*_G(\mathcal{W}^{s-1}_\mathcal{U})$.*

For the filtration of $H^*_G(X)$ defined above

$$H^*_G(X) = H^*_{G,0}(X) \supset H^*_{G,1}(X) \supset \cdots \supset H^*_{G,s}(X) \supset \cdots , \tag{6}$$

$H^*_G(X)$ is a filtered ring in the sense that

$$H^*_{G,s}(X) \cdot H^*_{G,s'}(X) \subset H^*_{G,s+s'}(X) , \tag{7}$$

thus $H^*_{G,s}(X)$ is an ideal in $H^*_G(X)$ (compare with [17, pages 145-146]).

Moreover, we have the following

Proposition 2.13. (i) *An element of $H^*_G(X)$ is in $H^*_{G,1}(X)$ if, and only if, its restriction to each orbit is zero, i.e.,*

$$H^*_{G,1}(X) = \ker(\, H^*_G(X) \to \prod_{x\in X} H^*_G(G/G_x)\,) = \bigcap_{x\in X} \ker(\, H^*_G(X) \to H^*_G(G/G_x)\,) .$$

Proof. A proof is analogous to the proof of [17, Proposition 5.1(i), page 146]. □

Lemma 2.14. *If X is a compact G-space such that* $\dim X/G \le r$, *then* $(H^*_{G,1}(X))^r = 0$.

If X is a compact G-space such that $\dim X/G \le 2r-1$, *then* $(H^*_{G,2}(X))^r = 0$.

Proof. It follows from Definition 2.12 and properties (6) and (7) of the filtration of $H^*_G(X)$,

$$H^*_G(X) = H^*_{G,0}(X) \supset H^*_{G,1}(X) \supset \cdots \supset H^*_{G,s}(X) \supset \cdots ,$$

that $H^*_{G,s}(X) = 0$, for all $s > \bar{s}$, where $\bar{s} = \dim X/G$. Since by assumption $\bar{s} = \dim X/G \le r$, in particular, we have that $H^*_{G,r}(X) = 0$. Consequently, if $\dim X/G \le 2r-1$ then $(K^*_{G,2}(X))^r \subset H^*_{G,2r}(X)$, and we conclude that $(H^*_{G,2}(X))^r = 0$. □

Remark 2.15. From [7, Theorem 1.1], it follows that for an action of a finite group G on a paracompact Hausdorff space X, we have $\dim X = \dim X/G$. Consequently, in this case, to estimate $\dim X$ it is enough to estimate $\dim X/G$.

Lemma 2.16. *Let X be a compact G-space $G = \mathbb{Z}_2^k$, or $G = \mathbb{Z}_p^k$, and $p_X : X \to \mathrm{pt}$ the canonical map. Let H be a maximal subgroup of G, $X^H \overset{\iota}{\subset} X$, the corresponding G-invariant subspace and $p_{X^H} : X^H \to \mathrm{pt}$, $p_{X^H} = p_X\,\iota$ the corresponding map. Finally, let $A = G/H \in \mathcal{A}$ be the orbit corresponding to H.*

i) *If $\omega \in I = (w_1, w_2, \dots, w_k) \cap \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H)) \subset R = F_2[w_1, w_2, \dots, w_k]$, then $p_X^*(\omega) \in H_{G,1}^*(X^H)$.*

ii) *If $\omega \in I = (c_1, c_2, \dots, c_k) \cap \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H)) \subset R = F_p[c_1, c_2, \dots, c_k] \otimes_{F_p} E(w_1, w_2, \dots, w_k)$, then $p_X^*(\omega) \in H_{G,2}^*(X^H)$.*

Proof. First we show that the description of $H_{G,1}^*(X)$ of Proposition 2.13 can be simplified as follow:

$$\begin{aligned} H_{G,1}^*(X) &= \ker(\, H_G^*(X) \to \prod_\alpha \prod_{x\in X^{H_\alpha}} H_G^*(G/G_x)\,) \\ &= \bigcap_\alpha \bigcap_{x\in X^{H^\alpha}} \ker(\, H_G^*(X) \to H_G^*(G/G_x)\,)\,, \end{aligned} \tag{8}$$

where the product, or intersection, are taken over all maximal isotropy groups H_α in X, i.e. all maximal subgroups H_α of G for which $X^{H_\alpha} \neq \emptyset$. Indeed, since for every $x \in X$, $X^{H_\alpha} \subset X^{G_x}$ for some maximal H_α, with $G_x \subset H_\alpha$, and $G/G_x \to G/H_\alpha$ is a G-map, we have

$$\ker(\, H_G^*(X) \to H_G^*(G/H_\alpha)\,) \subset \ker(\, H_G^*(X) \to H_G^*(G/G_x)\,).$$

If $\omega \in \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/G_x))$, then $p_X^*(\omega) \in \ker(\beta^* : H_G^*(X) \to H_G^*(G/G_x))$, for any G-map $\beta : G/G_x \to X$, in particular for the inclusion $Gx \subset X$ (see [3, page 59]).

Now, if $X = X^H$, H maximal, then for every $x \in X^H$, $G_x = H$ and the condition (8) reduces to $H_{G,1}^*(X) = \ker(\, H_G^*(X) \to H_G^*(G/G_{x_0})$ with any $x_0 \in X^H$. This shows the first case for $p = 2$.

Let p be an odd prime. Using the first part of statement, we have $p_{X^H}^*(\omega) \in H_{G,1}^*(X^H)$. To show that $p_{X^H}^*(\omega) \in H_{G,2}^*(X^H)$, we have to explore the geometric meaning of elements $c_i \in H_G^*(\mathrm{pt})$ and the space $\mathcal{W}$. First let us observe that the action of G on factorizes by the action of $G/H = \mathbb{Z}_p$, and the latter is free a $\mathbb{Z}_p$-space. Moreover, since the exact sequence $H \hookrightarrow G \to G/H$ splits, we have $G \simeq H \times \mathbb{Z}_p = H'$. Representing G as the product $\mathbb{Z}_p \times \mathbb{Z}_p \cdots \times \mathbb{Z}_p = H \times H'$, we introduce new coordinates, which are expressed by the old by an automorphism of G.

In such presentation we have $H^*(BG; F_p) = F_p[\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k] \otimes_{F_p} E(\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_k)$, $H_G^*(G/H; F_p) = H^*(BH; F_p) = F_p[\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_{k-1}] \otimes_{F_p} E(\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_{k-1})$ and

$$\ker H_G^*(\mathrm{pt}) \to H_G^*(G/H) = F_p[\tilde{c}_k] \otimes_{F_p} E(\tilde{w}_k) = H^*(B\mathbb{Z}_p; F_p).$$

Furthermore, the ideal $I = (c_1, c_2, \dots, c_k)$ is equal to the corresponding ideal

$$I = (\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_k).$$

In these coordinates, we have $I \cap \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H) = (\tilde{c}_k)$ is the principal ideal generated by $\tilde{c}_k$, shortly $\tilde{c}$. But $\tilde{c}$ is the first Chern class (reduces modulo p) of the universal

bundle ξ over $B\mathbb{Z}_p$ belongs to $H^2(B\mathbb{Z}_p; F_p)$. Consequently, $\tilde{c}$ vanishes on ${B\mathbb{Z}_p}^{(1)}$, the one skeleton of $B\mathbb{Z}_p$.

Turning back to the space $\mathcal{W}_{\mathcal{U}}$, associated with a G-space X and its G-covering $\mathcal{U}$. $\mathcal{W}_{\mathcal{U}}$ is the geometric realization of the nerve of $\mathcal{U}$, thus it is a G-complex G-homotopy equivalent to X. Furthermore, $\mathcal{U}$ induces a $\mathbb{Z}_p = G/H$-covering $\mathcal{U}^H$ of X^H, consisting of sets $U_i \cap X^H$. Next, $Y = \mathcal{W}^H$ is a free $\mathbb{Z}_p$-space and the map

$$p^* : (H^*(B\mathbb{Z}_p; F_p) = H^*_{\mathbb{Z}_p}(\mathrm{pt})) \to (H^*(Y \times_{\mathbb{Z}_p} E\mathbb{Z}_p; F_p) = H^*_{\mathbb{Z}_p}(Y; F_p))$$

is induced by the classifying map $\phi : Y \to E\mathbb{Z}_p$ of the $\mathbb{Z}_p$-principal bundle $\gamma = (Y, \pi, Y/\mathbb{Z}_p)$. Since $Y = \mathcal{W}^H$ is a $\mathbb{Z}_p$-complex, the map ϕ can be replaced by G-homotopic cellular map $\psi : Y \to E\mathbb{Z}_p$, e.g., $\psi(\mathcal{W}^{(1)}) \subset E\mathbb{Z}_p^{(1)}$. This means that $p_Y^*(\tilde{c}) = (\phi_Y/\mathbb{Z}_p)^*(\tilde{c})$ vanishes on $\mathcal{W}^{(1)}$, which shows that $p_Y^*(\tilde{c}) \in H^*_{G,2}(X^H)$ and it completes the proof. □

Now we are in position to prove Theorem 2.7.

Proof. (of Theorem 2.7). Denote by $\mathcal{U} = S(V) \backslash Z_f$, which is an open and invariant set. From the continuity of the equivariant Borel-Alexander-Spanier cohomology theory, it follows that there exists an open invariant set $\mathcal{V} \subset S(V)$ such that

$$Z_f \subset \mathcal{V} \text{ and } H^*_G(\mathcal{V}) = H^*_G(Z_f).$$

It yields $l(Z_f) = l(\mathcal{V})$. Moreover,

$$H^*_G(W \setminus \{0\}) = H^*_G(S(W)),$$

by the equivariant deformation argument and then $l(W \setminus \{0\}) = l(S(W))$. Since f maps equivariantly $\mathcal{U}$ into $W \setminus \{0\}$ we have

$$l(\mathcal{U}) \leq l(W \setminus \{0\}) = l(S(W)),$$

by the corresponding monotonicity property of the length index (cf. [3, Theorem 4.6]). Obviously, $\mathcal{U} \cup \mathcal{V} = S(V)$. It follows by the additivity property of the length index (cf. [3, Theorem 4.6]) that

$$l(S(V)) \leq l(\mathcal{V}) + l(\mathcal{U}),$$

which gives

$$l(Z_f) \geq l(S(V)) - l(S(W)).$$

By Propositions 2.10 and 2.11 we have $l(S(W)) = d(W)$. This gives

$$l(Z_f) \geq d(V) - d(W)\,. \tag{9}$$

Set $d(Z_f) := \sum_\alpha \dim(Z_f^{H_\alpha})$, where H_α are maximal isotropy groups in Z_f. To complete the proof of Theorem 2.7, it is enough to show that

$$l(Z_f) \leq d(Z_f) + 1, \tag{10}$$

if $p = 2$ and

$$2\,l(Z_f) \leq d(Z_f) + 1, \tag{11}$$

if p is an odd prime. Denote $r = d(Z_f)$, $X = Z_f$ and consider multiplies

$$p_X^*(\omega_1) \cdot p_X^*(\omega_2) \cdots p_X^*(\omega_r) \cdot p_X^*(\omega_{r+1}), \tag{12}$$

where $G = \mathbb{Z}_2^k$, and $\omega_i \in \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H_i))$, H_α a maximal subgroup of G.

For p an odd prime, we consider

$$p_X^*(\omega_1) \cdot p_X^*(\omega_2) \cdots \cdot p_X^*(\omega_t), \quad \text{where } \ 2\,t \geq r+1\,, \tag{13}$$

$\omega_i \in I \cap \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H_i))$, H_α a maximal maximal subgroup of G, $I = (c_1, c_2, \dots, c_k)$.

To show estimates (10) and (11), it is sufficient to show that the multiplies (12) and (13) are equal to zero.

Collecting ω_i into groups corresponding to a given maximal subgroup H_α, i.e. with $H_i = H_\alpha$. First note that there exists a H_α such that there are s factors in the multiple (12), with $s \geq \dim Z_f^{H_\alpha} + 1$. Analogously, there exists a H_α such that there are s factors in the multiple (13) corresponding to H_α, with $2s \geq \dim Z_f^{H_\alpha} + 1$.

By changing indices, we can assume that these factors are $p_X^*(\omega_1) \cdot p_X^*(\omega_2) \cdot \cdots \cdot p_X^*(\omega_s)$ in the both cases. By simplicity denote H_α by H.

Let $\iota_H : X^H \subset X$ be G-equivariant embedding. Now, we show that

$$\iota_H^*(p_X^*(\omega_1) \cdot p_X^*(\omega_2) \cdot \cdots \cdot p_X^*(\omega_s)) = 0 \tag{14}$$

in $H_G^*(X^H)$.

Indeed, put $\tilde{\omega}_i = \iota_H^*(p_X^*(\omega_i))$. Since $p_X^* \circ \iota_H^* = p_{X^H}^*$, $\tilde{\omega}_i$ is in the form $p_{X^H}^*(\omega_i)$ with $\omega_i \in \ker(H_G^*(\mathrm{pt}) \to H_G^*(G/H))$. If prime p is odd, then also $\omega_i \in I = (c_1, c_2, \dots, c_k)$, by the assumption on ω_i.

Now, applying Lemma 2.16 we have $\tilde{\omega}_i \in H_{G,1}^*(X^H)$, or $\tilde{\omega}_i \in H_{G,2}^*(X^H)$, if p is odd. Using the property (7) of filtration we see that

$$\tilde{\omega}_1 \cdot \tilde{\omega}_2 \cdot \cdots \cdot \tilde{\omega}_s \in H_{G,s}^*(X^H) = 0\,,$$

because $s \geq r+1$. This shows the equality (14), in general, e.g., also for $p = 2$.

If p is an odd prime, then additionally $\tilde{\omega}_i \in H_{G,2}^*(X^H)$ and consequently, by the property (7) of filtration we have

$$\tilde{\omega}_1 \cdot \tilde{\omega}_2 \cdot \cdots \cdot \tilde{\omega}_s \in H_{G,2s}^*(X^H) = 0\,,$$

because $2s \geq r+1$. This shows the equality (14) in the second case.

Recall, that our task is to show that the multiplies (12) (13) are equal to 0 in $H_G^*(X)$, but ι_H^* is not a monomorphism in general.

To prove it we use the Borel localization theorem (cf. [10] and [8]), which says that

$$S^{-1}\iota^* : S^{-1}H_G^*(X) \to S^{-1}H_G^*(X^H)$$

is an isomorphism, if we localize $H_G^*(X)$, and $H_G^*(X^H)$ in an appropriate multiplicative system $S \subset H_G^*(\mathrm{pt})$.

But, the localization operation $S^{-1} : H_G^*(X) \to S^{-1}H_G^*(X)$ is not a monomorphism, in general. However, the elements $p_X^*(\omega_i)$ are not annihilated by the used multiplicative system S. Let us fix a maximal subgroup $H \subset G = \mathbb{Z}_p^k$, $k \geq 2$. (One can consider that $H = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$, $k-1$ times, the sub-torus of rank $k-1$.)

We say that an element $\omega \in H_G^*(X)$ comes from X^H, if there exists $\tilde{\omega} \in H_G^*(X^H)$ such that $\iota_H^*(\omega) = \tilde{\omega}$.

Proposition 2.17. *For a given maximal subgroup $H \subset G$, $G = \mathbb{Z}_p^k$, with $X^H \neq \emptyset$, there exists a multiplicative system $S \subset H_G^*(\mathrm{pt}; F_p)$, contained in the center of $H_G^*(\mathrm{pt}; F_p)$ and such that:*

i) *The natural embedding* $\iota : X^H \to X$ *induces an isomorphism*

$$S^{-1}H^*_G(X;F_p) \xrightarrow{S^{-1}\iota^*_H} S^{-1}H^*_G(X^H;F_p)\,,$$

ii) *The canonical mapping* $H^*_G(X;F_p) \to S^{-1}H^*_G(X;F_p)$ *is injective on the set of elements that come from* X^H.

iii) *The canonical map* $H^*_G(X^H;F_p) \to S^{-1}H^*_G(X^H;F_p) = H^*_G(X^H;F_p) \otimes_{\mathbb{Z}_p} S^{-1}H^*_G(\mathrm{pt})$ *is injective.*

Proof. First, we can assume that $H = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \times e \subset \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \times \mathbb{Z}_p$, applying an automorphism $A : G \to G$, if necessary. Let $I = (w_1, \ldots, w_k)$, if $p = 2$ and $I = F_p[c_1, \ldots, c_k] \subset H^*_G(\mathrm{pt})$, if p is odd.

Then, $I \cap \ker(H^*_G(\mathrm{pt}) \to H^*_G(G/H))$, i.e., $\ker(H^*(BG,F_p) \to H^*(BH;F_p))$ is the ideal generated by w_k, or by c_k, respectively.

Take as the multiplicative system

$$S = F_2[w_1, \ldots, w_{k-1}] \setminus (w_k) \subset F_2[w_1, \ldots, w_{k-1}, w_k] \setminus \{0\} = H^*_{\mathbb{Z}_2^k}(\mathrm{pt}) \setminus \{0\},$$

if $p = 2$, or respectively,

$$S = F_p[c_1, \ldots, c_k] \setminus (c_k) \subset F_p[c_1, \ldots, c_{k-1}, c_k] \setminus \{0\} \subset H^*_{\mathbb{Z}_p^k}(\mathrm{pt}) \setminus \{0\},$$

if p is odd.

Note that S is a multiplicative system as the complement of a maximal ideal. In both cases, S is in the center of $H^*_G(\mathrm{pt})$, because in the first case the coefficients are F_2, and in the second, S contains only elements of even gradation (cf. [8] 3.3 page 190).

Next, we have to describe a family $\mathfrak{F}(S)$ of subgroups of G, which is associated with S by the following definition (cf. [8] 3.3 page 190)

$$\mathfrak{F}(S) = \{K : S \cap \ker(H^*_G(\mathrm{pt}) \to H^*_G(G/K)) \neq \emptyset\}. \tag{15}$$

We claim that in our case of H-maximal, $H = \mathbb{Z}_p \times \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p \times e$ we have

$$\mathfrak{F}(S) = \{K \subset G : K \neq H\,, H \neq G\}.$$

Each subgroup $K \subset G$, of rank $1 \leq r \leq k-1$, is the intersection of kernels of $k-r$ functionals $\phi^j_K : \mathbb{Z}_p^k \to \mathbb{Z}_p$, or equivalently, $\phi^j_K : F_p^k \to F_p$, i.e., ϕ^j_K is given as

$$\phi^j_K(x_1, x_2, \ldots x_k) = \sum_{i=1}^{k} a^j_i x^j_i,\ 1 \leq j \leq k-r,$$

for some $a^j_i \in \mathbb{Z}_p = F_p$.

Consequently, for $p = 2$, the ideal $\ker(H^*_G(\mathrm{pt}) \to H^*_G(G/K) \subset F_2[w_1, \ldots, w_{k-1}, w_k]$ is the ideal $(\phi^1_K, \ldots, \phi^{k-r}_K)$, and it has nonempty intersection with $S = F_2[w_1, \ldots, w_k] \setminus (w_k)$. Indeed, otherwise $(\phi^1_K, \ldots, \phi^{k-r}_K) \subset (w_k)$, which is impossible, since $K \neq H$, i.e., for at least one i, $1 \leq i \leq k-1$, we have $a^j_i \neq 0$.

By the same argument for p odd, the ideal

$$I \cap \ker(H^*_G(\mathrm{pt}) \to H^*_G(G/K) \subset F_p[c_1, \ldots, c_{k-1}, c_k]$$

is the the ideal $(\phi^1_K, \phi^2_K, \ldots, \phi^{k-r}_K)$, and it has nonempty intersection with

$$S = F_p[c_1, \ldots, c_k] \setminus (c_k).$$

Indeed, otherwise $(\phi_K^1, \dots, \phi_K^{k-r}) \subset (c_k)$, which is impossible, since $K \neq H$, i.e., for at least one $1 \leq j \leq k-r$, $1 \leq i \leq k-1$, we have $a_i \neq 0$.

Now, we can use the localization theorem ([8, Theorem 3.6]), which states that if S is a multiplicative system in $H_G^*(\mathrm{pt})$ and $\mathfrak{F}(S)$ the corresponding family of subgroups and $A \subset X$ an invariant closed and taut subset, then

$$S^{-1}H_G^*(X) \simeq S^{-1}H_G^*(A), \tag{16}$$

provided $X \setminus A$ is of finite S-type. Since we are considering metric spaces, the set $A = X^H$ is taut. Also, it is of finite S-type with respect to the above S, because in $X \setminus X^H$ all the isotropy groups belong to $\mathfrak{F}(S)$, $X \subset S(V)$ and the action is linear.

This shows i) of the statement of Proposition 2.17. Proofs of items ii) and iii) are consequence of i) and properties of the operation of localization S^{-1}, which is tensoring by the localized coefficient ring $S^{-1}H_G^*(\mathrm{pt}; F_p)$ (cf. [8, Chapter 3]). $\square$

As a direct consequence of Proposition 2.17, we get the following.

Corollary 2.18. *Under the notations and assumptions of Proposition 2.17. If $\eta_1, \eta_2, \dots, \eta_s$ are elements of $H_G^*(X; F_p)$ that come from X^H, then*

$$\iota^*(\eta_1 \cdot \eta_2 \cdot \dots \cdot \eta_s) = 0 \implies \eta_1 \cdot \eta_2 \cdot \dots \cdot \eta_s = 0\,.$$

Now, Corollary 2.18 applied to $X = Z_f$, $\eta_i = p_X^*(\omega_i)$ and combined with (14) shows that

$$p_X^*(\omega_1) \cdot p_X^*(\omega_2) \cdot \dots \cdot p_X^*(\omega_s) = 0 \text{ in } H_G^*(X; F_p).$$

This completes the proof of Theorem 2.7. $\square$

3. The case of $G = \mathbb{T}^k$

Now we formulate corresponding theorems for $G = \mathbb{T}^k$. If $X = S(V)$, V an orthogonal representation of $\mathbb{T}^k$ with $V^G = \{0\}$, then the maximal subgroups $G_x \subset G$ are more effectively described as in general case of arbitrary X, similarly as in the already discussed case $G = \mathbb{Z}_p^k$. Unfortunately, there is no one-one correspondence between fixed points of maximal subgroups and the irreducible representations that appear in the decomposition of V, because there are no maximal proper subgroups of this group.

Definition 3.1. *Let X be a G-space. We call an isotropy type (H), corresponding to an isotropy subgroup G_x of $x \in X$, the maximal isotropy type, if there is no $y \in X$ such that $G_x \subsetneq G_y$.*

If G is abelian, then we call $H = G_x$ the maximal isotropy subgroup.

If V is an orthogonal representation of G, then by a maximal isotropy type, correspondingly maximal isotropy subgroup H, if G is abelian, we mean the maximal isotropy subgroup, or respectively, the maximal isotropy subgroup of G, if G is abelian, of the action on the sphere $S(V)$.

Since, for a point direct sum of two representations V_1, V_2 we have

$$G_{(x_1,x_2)} = G_{x_1} \cap G_{x_2}\,,$$

a maximal isotropy subgroup of on orthogonal representation of $\mathbb{T}^k$ is the kernel group of an irreducible representation $V_\alpha \subset V$.

For an orthogonal representation V of $G = \mathbb{T}^k$ let

$$V = \bigoplus_{\alpha} l_\alpha V_\alpha = \bigoplus_{\alpha} V^\alpha, \quad V^\alpha := l_\alpha V_\alpha \tag{17}$$

be the decomposition of V into multiplies of irreducible representations. Every V_α is given by a homomorphism $\xi_\alpha : \mathbb{T}^k \to S^1$. Since V_α is not the trivial irreducible representation, ξ_α is onto and $H_\alpha = \ker \xi_\alpha \subset \mathbb{T}^k$ is a subgroup of $\mathbb{T}^k$ of dimension $k-1$.

Example 3.2. Let V be an orthogonal representation of $G = S^1$ with $V^G = \{0\}$. Then,

$$V = V_{m_1} \oplus V_{m_2} \oplus \cdots \oplus V_{m_r},$$

where V_{m_j} is a two-dimensional irreducible representation of S^1 given by the homomorphism $\xi_{m_j} : S^1 \to S^1$ defined as

$$t \mapsto \exp(2\pi \imath m_j t)\,.$$

Then, an isotropy subgroup $H = \mathbb{Z}_m \subset S^1$ is a maximal isotropy subgroup, if $m = m_j$, for some $1 \leq j \leq r$ and $m_j \nmid m_{j'}$, for some $j' \neq j$. In other words, maximal subgroups corresponds to all among m_j, which are not proper divisors of any other $m_{j'}$.

It is well-known, that any subgroup $H \subset \mathbb{T}^k$ is equal, up to isomorphism of $\mathbb{T}^k$, to $S^1 \times S^1 \times \cdots \times S^1 \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \cdots \times \mathbb{Z}_{m_r}$, for some $1 \leq r \leq k$ and $m_j \geq 1$.

Consequently, the kernel H_α of an irreducible representation, up to an isomorphism of $\mathbb{T}^k$, is equal to $S^1 \times S^1 \times \cdots S^1 \times \mathbb{Z}_n \subset \mathbb{T}^k$, i.e, it is equal to $H^0_\alpha \times \mathbb{Z}_{n_\alpha}$, where H^0_α is the component of identity of H_α. Such a subgroup of torus is called a subgroup of rank $k-1$.

Lemma 3.3. *If H_α and H_β are two different isotropy subgroups of rank $k-1$ of an orthogonal representation V of $G = \mathbb{T}^k$ of the form $H_\alpha = H^0_\alpha \times \mathbb{Z}_{m_\alpha}$ and $H_\beta = H^0_\beta \times \mathbb{Z}_{m_\beta}$, then*

i) *either $k \geq 2$ and $\dim(H_\alpha \cap H_\beta) = k-2$, and $V^{H_\alpha} \cap V^{H_\beta} = \{0\}$ in such a case,*
ii) *or $H^0_\alpha = H^0_\beta \simeq \mathbb{T}^{k-1}$ and if the lowest common multiple $n = [m_\alpha, m_\beta]$ of m_α and m_β and the induced action of S^1 on $V^{H^0_\alpha}$ we have $(V^{H^0_\alpha})^{\mathbb{Z}_n} = \{0\}$, then also $V^{H_\alpha} \cap V^{H_\beta} = \{0\}$.*

Proof. The dimension of the tangent space $T(H_\alpha) \subset TG \simeq \mathbb{R}^k$ is equal to $\dim H_\alpha = \dim H^0_\alpha = k-1$. If $\dim T(H_\alpha) \cap T(H_\beta) = k-2$, then two linear subspaces $T(H_\alpha)$ and $T(H_\beta)$ span TG, thus the group (H_α, H_β), generated by H_α and H_β is equal to G. Consequently, $V^{H_\alpha} \cap V^{H_\beta} = V^{(H_\alpha, H_\beta)} = \{0\}$ which shows the first case of statement.

Since G is abelian, all the subgroups are normal, e.g, for every $H \subset G$ the subspace V^H is a sub-representation of V. For $H = H_0 \times \mathbb{Z}_m$, we have $V^H = (V^{H^0})^{\mathbb{Z}_m}$ and the action of G on V^{H^0} factorizes through the factor $S^1 = \mathbb{T}^k / H^0$.

Let $H_\alpha, H_\beta \subset G$ be two subgroups isomorphic to $H^0 \times \mathbb{Z}_{m_\alpha}$, where H^0 is a $k-1$-dimensional torus (note that each of them is equal to $H^0 \times \mathbb{Z}_m$ but, up to a isomorphism, which can be different in each case). Using the fact that $V^{H_\alpha} = (V^{H^0_\alpha})^{H_\alpha / H^0_\alpha} = (V^{H^0_\alpha})^{\mathbb{Z}_{m_\alpha}}$, and analogously, for V^{H_β} we see that

$$V^{H_\alpha} = \bigoplus_{m_\alpha | m} V_m \quad \text{and} \quad V^{H_\beta} = \bigoplus_{m_\beta | m} V_m$$

for the induced action of $S^1 = G/H^0$. Consequently, if $(m_\alpha, m_\beta) = 1$ then there is no a common summand in the above direct sums and, consequently, $V^{H_\alpha} \cap V^{H_\beta} = \{0\}$ $\square$

It is worth pointing out that the fixed points subspaces of maximal isotropy groups do not span the entire representation V as in the case of $G = \mathbb{Z}_p^k$.

Example 3.4. *As an example let us take $G = S^1$ and its orthogonal representation $V = V_5 \oplus V_{15} \oplus V_3 \oplus V_{21} \oplus V_7$. Then minimal isotropy subgroups are $\mathbb{Z}_{15}$ and $\mathbb{Z}_{21}$. We have $V^{\mathbb{Z}_{15}} = V_{15}$ and $V^{\mathbb{Z}_{21}} = V_{21}$, which do not span V.*

On the other hand for $X = S(V)$ the maximal isotropy subgroups of rank $k-1$ always appear as the isotropy groups of $x \in S(V_\alpha)$. The above leads to the following.

For a given isotropy subgroup $H = H^0 \times \mathbb{Z}_m$ of rank $k-1$, we denote by $\mathcal{H}$ the family of all isotropy subgroups which the connected component of identity is equal to H^0.

By $V^{\mathcal{H}}$ we denote the sum

$$\sum_{H_\alpha \in \mathcal{H}} V^H = \bigoplus_{H_\alpha \in \mathcal{H}} V^\alpha, \tag{18}$$

which is a G-invariant linear subspace, thus a sub-representation. Here, H_α is the kernel group of irreducible representation V_α.

Lemma 3.5. *Let V be an orthogonal representation of $G = \mathbb{T}^k$. Then, V decomposes as the direct sum*

$$V = \bigoplus_{\mathcal{H}} V^{\mathcal{H}},$$

where the summation ranks all the isotropy subgroups of rank $k-1$.

Moreover, the action of $G = \mathbb{T}^k$ on $V^{\mathcal{H}}$ is given by the action of quotient group $S^1 = G/H^0$.

Proof. Since two different tori of rank $k-1$ in $\mathbb{T}^k$ intersects along a torus of dimension $k-2$, the subspaces $V^{\mathcal{H}}$ and $V^{\mathcal{K}}$ are perpendicular, if $\mathcal{H} \neq \mathcal{K}$, as follows from Lemma 3.3 i).

On the other hand, by (18) the spaces $V^{\mathcal{H}}$ span V.

Since $H^0 \subset H$ implies $V^H \subset V^{H^0}$, for every $H \in \mathcal{H}$, the $k-1$-dimensional torus $H^0 = \mathbb{T}^{k-1}$ acts on $V^{\mathcal{H}}$ trivially. Consequently, the action of $G = \mathbb{T}^k$ on $V^{\mathcal{H}}$ is induced by the action of $S^1 = G/H^0$. □

We denote by $d_{\mathcal{H}}(V)$ the complex dimension $\dim_{\mathbb{C}} V^{\mathcal{H}}$.

We summarize the above results as follow.

Fact 3.6. *For every orthogonal representation of $G = \mathbb{T}^k$, with $V^G = \{0\}$ we have the canonical decomposition into irreducible representations, each of them possessing a complex structure consistent with the action and independently into the orthogonal direct sum of sub-representations of fixed points of all $H \in \mathcal{H}$,*

$$V = \bigoplus_{\alpha} l_\alpha V_\alpha = \bigoplus_{\alpha} V^\alpha = \bigoplus_{\mathcal{H}} V^{\mathcal{H}},$$

where at the latter ranks all distinct families $\mathcal{H}$ of isotropy subgroups of G on $S(V)$ with given $k-1$ torus $H^0 \subset \mathbb{T}^k$ being the component of identity.

Consequently,

$$\sum_{\alpha} l_\alpha = d(V) = \sum_{\mathcal{H}} d_{\mathcal{H}}(V)$$

and the spheres $S(V^\alpha) = S(V^{H_\alpha})$ form a family of disjoint invariants subsets of $S(V)$.

Suppose that $f : S(V) \to W$ is a G-equivariant map. Then, $f(S(V)^{\mathcal{H}}) \subset W^{\mathcal{H}}$ for every family (H) of the isotropy subgroups with a fixed connected component of the identity $H_0 \simeq \mathbb{T}^{k-1}$. Indeed, $f(S(V)^H) \subset W^H$, for every $H \in \mathcal{H}$.

Now, we discuss the induced $S^1 = \mathbb{T}^k/H^0$-equivariant map $f^{\mathcal{H}} : S(V)^{\mathcal{H}} \to W^{\mathcal{H}}$ for each family $\mathcal{H}$ to obtain an analogous of the Bourgin-Yang Theorem 2.3 for the torus.

Let V be an orthogonal representation of $G = S^1$ such that $V^G = \{0\}$. Then, $V = V_{m_1} \oplus V_{m_2} \oplus \cdots \oplus V_{m_s}$, $m_j \in \mathbb{N}$, $1 \leq j \leq s$.

Consider the subset $\mathcal{M}(V)$ of $\mathbb{N}$ consisting of all m_j and their divisors, called *the set of periods* of V. We have a natural semi-order in $\mathcal{M}(V)$ given by

$$n \preccurlyeq m \iff n \mid m \tag{19}$$

We distinguish the subset $\{m_1, m_2, \ldots, m_s\}$ of $\mathcal{M}(V)$ consisting of m_j as above, and called *the set of essential periods* of V, or generators of $\mathcal{M}(V)$, and denoted by $\mathcal{M}_{es}(V)$.

Note that the *maximal isotropy subgroups* of the action of S^1 on $S(V)$ corresponds to the maximal elements of $\mathcal{M}(V)$. In this case, the isotropy group on $S(V_{m_j})$ is equal to $\mathbb{Z}_{m_j}$, which corresponds to the ideal (m_j) of $\mathbb{Z}$. Of course maximal element of $\mathcal{M}(V)$ are essential periods, but not conversely.

In the set $\mathcal{M}(V)$ we can consider a natural filtration defined as follows.

Let $\mathcal{M}(V)^{(0)} = \mathcal{M}(V)_{(0)} = \{\bar{m}_j : \bar{m}_j \text{ is maximal in } \mathcal{M}(V)\}$. Let us define $\mathcal{M}(V)^1 = \mathcal{M}(V) \setminus \mathcal{M}(V)^{(0)}$ and subsequently let $\mathcal{M}(V)_{(1)} = \{m \in \mathcal{M}(V)^1 : m \text{ is maximal in } \mathcal{M}(V)^1\}$.

Thereafter put $\mathcal{M}(V)^{(1)} = \mathcal{M}(V)^{(0)} \cup \mathcal{M}(V)_{(1)}$.

In this way, we obtain an ascending filtration

$$\mathcal{M}(V)^{(0)} \subset \mathcal{M}(V)^{(1)} \cdots \subset \mathcal{M}(V)^{(r)} = \mathcal{M}(V)$$

and disjoint partition

$$\mathcal{M}(V)_{(0)} \cup \mathcal{M}(V)_{(1)} \cup \ldots \cup \mathcal{M}(V)_{(r)} = \mathcal{M}(V)\,.$$

The above defined filtration induces an ascending filtration, and correspondingly a partition, of $\mathcal{M}_{es}(V)$.

Moreover, the isotropy subgroup of a point $(x, y) \in S(l_j V_{m_j} \oplus l_i V_{m_i})$, $x \neq 0$, $y \neq 0$ is isomorphic to $\mathbb{Z}_{m_j} \cap \mathbb{Z}_{m_i} \subset S^1$ which is equal to $\mathbb{Z}_{(m_j, m_i)}$. Consequently, $V_{m_j} \oplus V_{m_i} \subset V^{\mathbb{Z}_{(m_j, m_i)}}$. This means that for every $m \in \mathcal{M}(V)$ the cyclic group $\mathbb{Z}_m$ is the isotropy subgroup of some $x \in S(V)$ and

$$V^{\mathbb{Z}_m} = \bigoplus_{m \mid m_j} l_j V_{m_j}\,.$$

In particular, $V = V^{\mathbb{Z}_{\bar{m}}}$ where $\bar{m} = (m_1, m_2, \ldots, m_s)$ is the unique minimal element of $\mathcal{M}(V)$.

We are in position to formulate our main theorem, describing an obstruction to the existence of S^1-equivariant map between the spheres of two orthogonal representations of S^1.

Theorem 3.7. *Let V, W be two orthogonal representations of $G = S^1$ such that $V^G = W^G = \{0\}$.*

Then, there exists a G-equivariant map $f : S(V) \to W \setminus \{0\}$ if and only if

$$\text{for every } m \in \{m_1, m_2, \ldots, m_s\} = \mathcal{M}_{es}(V) \text{ we have, } \; d(V^{\mathbb{Z}_m}) \leq d(W^{\mathbb{Z}_m})\,. \tag{20}$$

Moreover,

$$\dim Z_f \geq \max_{m \in \mathcal{M}(V)} \dim Z_f^{\mathbb{Z}_m} \geq 2\,(\,d(V^{\mathbb{Z}_m}) - d(W^{\mathbb{Z}_m})\,) - 1 \geq 2\,(\,d(V) - d(W)\,) - 1\,.$$

Note that if $\dim Z_f \geq -k$, for $k \in \mathbb{N}$, then there is no any condition on Z_f, i.e. it can be empty as well.

Corollary 3.8. [H. Hopf] *There is no a S^1-equivariant map $f : S(V) \to W \setminus \{0\}$, if $d(V) > d(W)$.*

Proof. (of Corollary 3.8) Note that if $d(V) > d(W)$, then the set $\{m \in \mathcal{M}(V) : d(V^{\mathbb{Z}_m}) > d(W^{\mathbb{Z}_m})\}$ is nonempty, because $\bar{m} = (m_1, m_2, \ldots, m_s) \in \mathcal{M}(V)$ and $V^{\mathbb{Z}_{\bar{m}}} = V$. Now the statement follows from Theorem 3.7 (see [12] for further references and historical remarks). □

Proof. (of Theorem 3.7) Let $G = S^1$, $V^G = W^G = \{0\}$, and $f : S(V) \to W$, be an S^1-equivariant map. For every $\mathbb{Z}_m \subset S^1$, the linear subspaces $V^{\mathbb{Z}_m}$ and $W^{\mathbb{Z}_m}$ are G-invariant, thus sub-representations with the induced action of $S^1 = S^1/\mathbb{Z}_m$ has no fixed points on the spheres.

Moreover, $f^{\mathbb{Z}_m} : S(V^{\mathbb{Z}_m}) \to W^{\mathbb{Z}_m}$ is S^1-equivariant, thus H-equivariant. Let us take a prime p not dividing m, i.e. relatively prime to m. Observe that $S(V^{\mathbb{Z}_m})$ and $S(W^{\mathbb{Z}_m})$ are free $\mathbb{Z}_p$-spaces. Now, it is enough to use Corollary 2.5, with $k = 1$ and $H = e$, to get

$$d(V^{\mathbb{Z}_m}) \leq d(W^{\mathbb{Z}_m})\,,$$

which shows that our condition is necessary.

Furthermore, if $m \in \mathcal{M}$ is such that $d(V^{\mathbb{Z}_m}) > d(W^{\mathbb{Z}_m})$ then for every S^1-equivariant map $f : S(V^{\mathbb{Z}_m}) \to W^{\mathbb{Z}_m}$ and a prime p as above, it follows from Theorem 2.3 applied to $f^{\mathbb{Z}_m}$, with $k = 1$ and $H = e$, that

$$\dim Z_{f^{\mathbb{Z}_m}} \geq 2\,(\,d(V^{\mathbb{Z}_m}) - d(W^{\mathbb{Z}_m})\,) - 1.$$

Since $Z_{f^{\mathbb{Z}_m}} \subset Z_f$, we get the left estimate of statement. Taking as m the minimal element $\bar{m} = (m_1, m_2, \,\ldots\,, m_s)$ we have

$$\dim Z_f = \dim Z_{f^{\mathbb{Z}_{\bar{m}}}} \geq 2\,(\,(d(V) - d(W^{\mathbb{Z}_{\bar{m}}})\,) - 1 \geq 2\,(\,d(V) - d(W)\,) - 1$$

which shows the right inequality.

We are left with the task to show that if the inequality $d(V^{\mathbb{Z}_m}) \leq d(W^{\mathbb{Z}_m})$ is satisfied for all $m \in \mathcal{M}_{es}(V)$, then there exists an S^1-equivariant map $f : S(V) \to S(W)$. We show it by a finite induction over the semi-order $\preccurlyeq$ in $\mathcal{M}(V)$, starting from maximal elements.

First suppose $m_0 \in \mathcal{M}(V)_{(0)}$ is a maximal element of $\mathcal{M}(V)$. Then $V^{\mathbb{Z}_{m_0}} = l_{m_0} V_{m_0}$, since m_0 is maximal, i.e. $\mathbb{Z}_{m_0}$ is a maximal isotropy group of the action of S^1 on $V \setminus \{0\}$. By definition, $d(V^{\mathbb{Z}_{m_0}}) = l_{m_0}$. On the other hand, by our assumption, $d(V^{\mathbb{Z}_{m_0}}) \leq d(W^{\mathbb{Z}_{m_0}})$. In W the subgroup $\mathbb{Z}_{m_0}$ is not maximal, in general, but $W^{\mathbb{Z}_{m_0}} = \tilde{l}_0 V_{m_0} \oplus \tilde{l}_1 V_{m_1} \oplus \cdots \oplus \tilde{l}_s V_{\mathbb{Z}_{m_s}}$, where $m_0 \mid m_j$, for every $1 \leq j \leq s$, i.e m_0 is not maximal in W, in general. But

$$d(W^{\mathbb{Z}_{m_0}}) \;=\; \sum_{j=0}^{s} \tilde{l}_j\,. \tag{21}$$

Since $l_{m_0} \leq \sum \tilde{l}_j$, there exists a S^1-equivariant map $\phi : S(l_{m_0} V_{m_0}) \to S(\overset{s}{\underset{j=0}{\oplus}} \tilde{l}_j\, V_{m_j})$, provided $m_0 \mid m_j$, for every j. Indeed, let s' be smallest number such that $l_{m_0} \leq \sum_{j=0}^{s'} \tilde{l}_j$. It is enough to map each $S(\tilde{l}_j V_{m_0})$ into $S(\tilde{l}_j V_{m_j})$ by the mapping $z \mapsto z^{\frac{m_j}{m_0}}$, for each $0 \leq j \leq s'$. If $l_{m_0} < \sum_{j=0}^{s'} \tilde{l}_j$, then for $j = s'$, we compose the above map with the embedding of $S(\bar{l}_{s'} V_{m_0}) \subset S(\tilde{l}_{s'} V_{m_0})$, where $\bar{l}_{s'} = \sum_{j=1}^{s'} \tilde{l}_j - l_{m_0}$. Finally, the joint of maps gives the required equivariant map from

$$\phi : S(l_{m_0} V_{m_0}) \to S(\overset{s'}{\underset{j=1}{\oplus}} \tilde{l}_j V_{m_j}) \subset S(\overset{s}{\underset{j=1}{\oplus}} \tilde{l}_j\, V_{m_j})\,.$$

Note that l_{m_0} can be greater than $\tilde{l}_0$, in general.

In this way, we define a G-equivariant map from $\cup S(V^{m_i}) \to S(W)$, where the union is taken over all $m_i \in \mathcal{M}_{(0)}(V)$ (note that $S(V^{m_i}) \cap S(V^{m_j}) = \emptyset$, if $m_i \neq m_j$ are maximals). Furthermore, applying the join construction, we construct a G-equivariant map $\phi^{(0)} : S(\oplus\, l_j V_{m_j}) \to S(W)$, where the sum is taken over all $m_j \in \mathcal{M}_{(0)}(V) = \mathcal{M}^{(0)}(V)$.

Let $m \in \mathcal{M}(V)_{(1)}$. If m is not essential, then

$$V^{\mathbb{Z}_m} = \underset{m|m_j, m_j \in \mathcal{M}(V)^{(0)}}{\oplus} l_j\, V_{m_j} \subset \underset{m_j \in \mathcal{M}(V)^{(0)}}{\oplus} l_j\, V_{m_j}\,.$$

Consequently, the equivariant map $\phi = \phi^{(0)} : S(V^{\mathbb{Z}_m}) \to W \setminus \{0\}$ is already defined in the previous step.

Suppose that $m \in \mathcal{M}_{es}(V)_{(1)}$ is essential, then

$$V^{\mathbb{Z}_m} = \underset{m|m_j, m_j \in \mathcal{M}(V)}{\oplus} l_j\, V_{m_j} = l_m\, V_m \oplus \underset{m|m_j, m_j \neq m, m_j \in \mathcal{M}(V)^{(0)}}{\oplus} l_j\, V_{m_j}\,.$$

Respectively,

$$W^{\mathbb{Z}_m} = \underset{m|m_j}{\oplus} \tilde{l}_j\, V_{m_j} = \tilde{l}_m V_m \oplus \underset{m|m_j, m_j \neq m}{\oplus} \tilde{l}_j\, V_{m_j}\,.$$

Now using the fact that

$$d(V^{\mathbb{Z}_m}) = l_m + \sum_{m|m_j, m_j \neq m, m_j \in \mathcal{M}(V)^{(0)}} l_j \leq \sum_{m|m_j} \tilde{l}_j = d(W^{\mathbb{Z}_m})\,,$$

and proceeding by the same way as in the first inductive step, we construct an equivariant map $\phi : S(V^{\mathbb{Z}_m}) \to W \setminus \{0\}$. Repeating the same for all $m \in \mathcal{M}_{es}(V)_{(1)}$ we construct an equivariant map

$$\phi^{(1)} : \underset{m \in \mathcal{M}(V)^{(1)}}{\cup} S(V^{\mathbb{Z}_m}) \to W \setminus \{0\}\,.$$

Now repeating all the procedure for $m \in \mathcal{M}_{(i)}$, with consecutive $2 \leq i \leq r$, we extend our G-equivariant map $\phi^{(1)}$ onto consecutive sets

$$\underset{m \in \mathcal{M}(V)^{(i)}}{\cup} S(V^{\mathbb{Z}_m})$$

and finally a G-equivariant map

$$\phi = \phi^{(r)} : \bigcup_{m \in \mathcal{M}(V)^{(r)}} S(V^{\mathbb{Z}_m}) \to W \setminus \{0\}$$

But

$$S\left(\bigcup_{m \in \mathcal{M}(V)^{(r)}} S(V^{\mathbb{Z}_m}) \right) = S(V^{\mathbb{Z}_{\bar{m}}}) = S(V)\,,$$

where $\bar{m} = (m_1, m_2 \ldots, m_s)$ is the greatest common divisor of $m_j \in \mathcal{M}(V)$. This defines a G-equivariant map

$$\phi = \phi^{(r)} : S(V) \to W \setminus \{0\}$$

and, consequently, completes the proof of Theorem 3.7.

Note that in our procedure of construction of ϕ we have to use the condition $d(V^{\mathbb{Z}_m}) \leq d(W^{\mathbb{Z}_m})$ of Theorem 3.7 only if m is an essential period.

□

Since for $G = S^1$ we have $\dim Z_f - \dim Z_f/G \leq 1 = \dim G$, we get the following consequence of Theorem 3.7.

Corollary 3.9. *Under the assumption of Theorem 3.7 we have*

$$\dim Z_f/G \geq 2\,(\,d(V) - d(W^{\mathbb{Z}_{\bar{m}}})\, - 1\,) \geq 2\,(\,d(V) - d(W)\, - 1\,)$$

As a consequence of Theorem 3.7, we obtain the following version of the Bourgin-Yang theorem for the torus.

Theorem 3.10. *Let V, W be two orthogonal representations of the group $G = \mathbb{T}^k$, with $V^G = W^G = \{0\}$, $V = \bigoplus_{\alpha} l_\alpha(V)\, V_\alpha = \bigoplus_{\mathcal{H}} V^{\mathcal{H}}$ and $W = \bigoplus_{\beta} l_\beta(W)\, V_\beta = \bigoplus_{\mathcal{H}} W^{\mathcal{H}}$ be the canonical decompositions of* (1) *and Fact* 3.6 *of V and W, respectively, where the families $\mathcal{H}$ are specified by the rank $k-1$ of the isotropy subgroups on $S(V)$. Then, there exists an equivariant map $f : S(V) \to W \setminus \{0\}$, if and only if, for every $\mathcal{H}$ we have*

$$d(V^{\mathcal{H}}) \leq d(W^{\mathcal{H}})\,,$$

and for the induced action of S^1 on $V^{\mathcal{H}}$, $W^{\mathcal{H}}$, and the S^1-equivariant map $f^{\mathcal{H}} : S(V^{\mathcal{H}}) \to W^{\mathcal{H}}$ induced by restriction, the corresponding condition (20) *of Theorem 3.7 is satisfied.*

If there exists family $\mathcal{H}$ such that $d(V^{\mathcal{H}}) > d(W^{\mathcal{H}})$, then

$$\dim Z_f \geq \dim(Z_{f^{\mathcal{H}}}) \geq 2\,(\,d(V^{\mathcal{H}}) - d(W^{\mathcal{H}})\,) - 1\,.$$

Furthermore

$$\dim Z_f \geq 2\,(\,d(V) - d(W)\,) - 1\,.$$

Corollary 3.11. *If $d(W) < d(V)$, then there is no G-equivariant map from $S(V)$ into $W \setminus \{0\}$.*

Proof. From condition (20), it follows that there exists a S^1-equivariant map from $S(V^{\mathcal{H}})$ into $W^{\mathcal{H}}$ for every family $\mathcal{H}$, which is G-equivariant by the form of the action of G on $V^{\mathcal{H}}$. Since $V = \bigoplus_{\mathcal{H}} V^{\mathcal{H}}$ (cf. Fact 3.6) we can join such maps as in the proof of Corollary 2.5 to get the required G-equivariant map. This shows the first part of statement.

If $d(V) > d(W)$, then $d(V^{\mathcal{H}}) > d(W^{\mathcal{H}})$, for at least one family $\mathcal{H}$. Now the second part of the statement follows from Theorem 3.7, or Corollary 3.8, applied to $f^{\mathcal{H}}$.

We show that there exists a subgroup $K \subset G$, $K \simeq S^1$, such that $V^K = \{0\}$. Since $S(V)$ is compact, there are only finitely many isotropy subgroups, each of them contained in an element of some $\mathcal{H}$. Now, we can find one-dimensional subtorus $K = S^1 \subset G$ such that $K \cap H_i^0 = e$, for $1 \leq i \leq s$. Indeed, there exists a line $L \subset T(\mathbb{T}^k)_e = \mathbb{R}^k$ such that:

- $L \cap T(H_{m_i}^0) = \{0\}$, for $1 \leq i \leq s$,
- For the integral lattice $\mathbb{Z}^k$ we have $L \cap \mathbb{Z}^k \neq \{0\}$, because the intersections of lines, satisfying the second condition with the unit sphere, are dense. It is clear that for $K = \exp(L)$ we have $S(V)^K = \emptyset$. (But the action of K is not free, in general). Now, the third part of the statement follows from Theorem 3.7, applied to the K-equivariant map $f : S(V) \to W$, where V and W are considered as representations of S^1. □

Theorem 3.12. *Let G be the circle S^1. Then, for the sphere $S(V)$ of a infinite-dimensional fixed point free normed G-vector space (orthogonal representation) V, a finite dimensional orthogonal representation W of G such that $W^G = \{0\}$, and a G-equivariant map $f : S(V) \to W$ we have*

$$\dim Z_f \; = \; \infty \, .$$

Proof. Let $\tilde{V}(d) = l_{m_1} V_{m_1} \oplus l_{m_2} V_{m_2} \oplus \cdots \oplus l_{m_s} V_{m_s}$ be a sub-representation of V such that $d(\tilde{V}) = d$. Always we can find a prime p which is relatively prime to all m_j, $1 \leq j \leq s$, thus relatively prime to all $m \in \mathcal{M}(\tilde{V})$. By the argument of the proof of Theorem 3.7, or directly by its thesis applied to the restriction $f_{|S(\tilde{V})}$ of f to $S(\tilde{V})$, we have

$$\dim Z_f \geq \dim Z_{f_{|S(\tilde{V})}} \geq 2\,(\, d(\tilde{V}) - d(W)\,) - 1\, .$$

Passing with d to ∞ we get the statement. □

Theorem 3.13. *Let G be the torus $\mathbb{T}^k$, $k \geq 2$. Then, for the sphere $S(V)$ of an infinite-dimensional fixed point free normed G-vector space (orthogonal representation) V, a finite dimensional orthogonal representation W of G such that $W^G = \{0\}$, and a G-equivariant map $f : S(V) \to W$ we have*

$$\dim Z_f \; = \; \infty \, .$$

Proof. Using Theorem 3.10 a proof is analogous to that of Theorem 3.12. □

4. p-toral groups

In this section, we show that Theorems 2.6 and 3.13 can be extended on a larger class of groups called p-toral. The main result will be formulated analogous to [3, Theorem 3.1].

Definition 4.1. *A compact Lie group G is called p-toral if it is of the form of an extension*

$$1 \hookrightarrow \mathbb{T}^k \hookrightarrow G \to P \to 1,$$

where P is a finite p-group.

In this section, we will use the G-index of G-spaces defined by the Borel equivariant stable cohomotopy theory, i.e the theory

$$h_G^*(X, A) = \pi_s^*(X \times_G EG, A \times_G EG),$$

where π_s^* denotes the stable cohomotopy theory.

Following [3, 5.4)], as a family $\mathcal{B}$ of orbits defining value of this length index, we take

$$\mathcal{B} = \{G/H : H \subsetneq G\}.$$

Taking $I = h^*(pt)$, h_G^* and $\mathcal{B}$ as above, the value of the length index defined by the triple $\{\mathcal{B}, h_G^*, I\}$ at a pair of G-spaces (X, X') will be denote by $l(X, X')$.

Theorem 4.2 (Characterization of p-toral groups)**.**

a) *Let G be a p-toral group $1 \hookrightarrow \mathbb{T}^k \to G \to P \to 1$. Then, for the sphere $S(V)$ of an infinite-dimensional fixed point free G-Hilbert space (orthogonal representation) V and a finite dimensional orthogonal representation W of G, such that $W^G = \{0\}$, and a G-equivariant map $f : S(V) \to W$, we have*

$$\dim Z_f \;=\; l(Z_f) \;=\; \infty\,.$$

b) *If G is not p-toral, then there exist an infinite-dimensional fixed point free G-Hilbert space V, a finite dimensional representation W of G with $W^G = \{0\}$ and an equivariant map $f : S(V) \to W$ such that*

$$Z_f = \emptyset, \quad e.g. \;\; \dim Z_f = -1 < \infty\,.$$

Proof. The part b) follows directly from [2, Theorem 2)]. It states that for any not p-toral group and every orthogonal Hilbert representation V, $V^G = \{0\}$, there exist an orthogonal representation W, with $W^G = \{0\}$ and $\dim W < \infty$, and a G-map $f : S(V) \to W \setminus \{0\}$. Therefore $Z_f = \emptyset$, which proves part b).

To show a) we adapt the arguments of [4] and [2] exposed in an extended form in [3, Chapter 5]. First, we have the following

Proposition 4.3 ([3, 5.11, 5.12])**.** *For a p-group P and a contractible P-space X, we have $l(X) = \infty$.*

Note that we do not require $X^P = \emptyset$. Remind that if V is an infinite-dimensional Hilbert space then $X = S(V)$ is a metric G-space which is contractible, because $S(V)$ is homeomorphic to $\mathring{D}(V)$. Consequently, it follows from Proposition 4.3 that

$$l(S(V)) = \infty\,.$$

On the other hand, we have the following

Proposition 4.4 (cf. [3, 5.4])**.** *For every finite dimensional orthogonal representation W, with $W^G = \{0\}$, and any G-length index as above, we have $l(S(W)) < \infty$.*

Proof. Indeed, $S(W)$ is a compact G-space and each orbit of $S(W)$ can be mapped into some element (orbit) of $\mathcal{B}$. Now, the statement reduces to [3, Corollary 4.9 b)]. $\square$

Note that the statement in the last propositions holds for every equivariant cohomology theory h^*.

An essential step in our proof is the following result.

Lemma 4.5. *Let G be a finite group. If X is a finite dimensional metric G-space, with $X^G = \emptyset$, then $l(X) < \infty$.*

Proof. Since X is finite dimensional, so is X/G. Since G is finite, there is only a finite number of orbit types on X. Now, by the Mostow theorem (cf. [6, Theorem 10.1]), there exist a finite dimensional orthogonal representation V of G and a G-embedding $\iota : X \to V$ of X into V. Since $X^G = \emptyset$, we have $\iota(X) \cap V^G = \emptyset$. Consequently, composing ι with $p_0^\perp : V \to V_\perp^G$, the orthogonal projection of V onto the orthogonal complement of V^G, we get an equivariant map $\phi : X \to V_\perp^G \setminus \{0\}$. Now, composing ϕ with the retraction $V_\perp^G \setminus \{0\} \to S(V_\perp^G)$ we obtain a G-equivariant map $\psi : X \to S(V_\perp^G)$. Therefore, it follows from the monotonicity property of the length index l (cf. [3, Theorem 4.7]) and Proposition 4.4 that

$$l(X) \le l(S(V_\perp^G)) < \infty.$$

□

Proof. (of Theorem 4.2 a))Suppose first that G is finite, i.e. $k = 0$, and $G = P$ is a finite p-group and let $f : S(V) \to W$ be a P-equivariant map. Since Z_f is closed G-invariant subspace of $S(V)$, by continuity property of the length index (cf. [3, 4.7 Continuity]), there exists an open P-invariant neighborhood $Z_f \subset \mathcal{U}$ such that $l(Z_f) = l(\mathcal{U})$.

Denote $\mathcal{V} = S(V) \setminus Z_f$ which is a P-invariant open subset of $S(V)$. Note that $f : \mathcal{V} \to W \setminus \{0\}$ is an equivariant map, and it follows from the monotonicity property of the index that $l(\mathcal{V}) \le l(W \setminus \{0\})$. Also, by Proposition 4.3, we have $l(S(V)) = \infty$, and $l(S(W)) < \infty$, by Proposition 4.4 and the assumption $W^G = \{0\}$.

On the other hand $W \setminus \{0\}$ is P-equivariantly homotopy equivalent to $S(W)$ and, consequently, has the same index. Using the subadditivity property of the index, we get

$$\infty \;=\; l(S(V)) \;\le\; l(\mathcal{U}) + l(\mathcal{V}) \le l(Z_f) + l(S(W)).$$

Since $l(S(W)) < \infty$, we conclude that $l(Z_f) = \infty$. Note that $Z_f^G = \emptyset$, because $S(V)^G = \emptyset$. If $\dim Z_f < \infty$, then $l(Z_f) < \infty$ by Lemma 4.5, which is a contradiction. Thus, $\dim Z_f = \infty$.

Now, assume that G is an extension $1 \to \mathbb{T}^k \hookrightarrow G \to P \to 1$, with $k \ge 1$. We distinguish two cases:

$$\text{either } \dim V^{\mathbb{T}^k} = \infty \,,\text{ or } \dim V^{\mathbb{T}^k} < \infty\,.$$

First suppose that $\dim V^{\mathbb{T}^k} = \infty$. Note that $V^{\mathbb{T}^k}$ has a natural action of $P = G/\mathbb{T}^k$, with the fixed point set $V^G = (V^{\mathbb{T}^k})^P = \{0\}$. Moreover, the restriction $f_{|S(V)^{\mathbb{T}^k}}$ maps $S(V)^{\mathbb{T}^k}$ P-equivariantly into $W^{\mathbb{T}^k} \subset W$. Applying the previous case for $G = P$, and for the triple $(V^{\mathbb{T}^k}, W^{\mathbb{T}^k}, f_{|S(V)^{\mathbb{T}^k}})$ we conclude that

$$\dim Z_f \ge \dim Z_{f^{\mathbb{T}^k}} = \infty\,.$$

Now assume that $\dim V^{\mathbb{T}^k} < \infty$. First observe that $V^{\mathbb{T}^k}$ is a $N(\mathbb{T}^k)$ invariant subspace of V, where $N(H)$ is the normalizer of H in G. But $\mathbb{T}^k$ is a normal subgroup as the component of the identity, thus $V^{\mathbb{T}^k}$ is a sub-representation of V. Let $V' = V_\perp^{\mathbb{T}^k}$ be the orthogonal complement of $V^{\mathbb{T}^k}$. By our assumption $\dim V' = \infty$.

Following a standard argument also used in [3, Proof of Theorem 3.1a]), we claim that for any p-toral group G and for every compact G-space A, with $A^G = \emptyset$, there exists a finite p-group P of G, which acts on A without fixed points: $A^P = \emptyset$. To see this, observe

that G can be approximated by finite p-groups. More precisely, for any natural number s consider the set $P^s := \{g \in G : g^{p^s} = e\}$. If p^s is a multiple of the order of G/G_0, G_0 the component of the identity, then P^s is a subgroup of G, according to known results (see [3, 5.4] for references). P^s is obviously a finite p-group which is an extension of $G_0 \cap P^s$ by G/G_0, i.e. has the form $1 \to P_s \hookrightarrow P^s \to P$ with $P_s = P^s \cap (G_0 = \mathbb{T}^k)$. Moreover, it is clear that $A^{P_s} = \emptyset$, if s is big enough, because A is a compact fixed point free G-space, i.e. has only a finite number of isotropy orbit types.

Now, take $A = S(W)$ and P^s as above. Then, $f : S(V') \to W$ is a P^s-equivariant map and $W^{P^s} = \{0\}$. Consider $\tilde{V} = (V')^{P^s}$. If $\dim \tilde{V} = \infty$, then $\dim Z_{f|S(\tilde{V})} = \dim S(\tilde{V}) = \infty$, because $f(S(V)^{P^s}) \subset W^{P^s} = \{0\}$. Consequently, assume that $\dim \tilde{V} < \infty$. Once more, $\tilde{V}$ is a P^s-subrepresentation and we can take the orthogonal complement V'' of it in V'. By our assumption and the choice of V'', we have $(S(V''))^{P^s} = \emptyset$ and $\dim V'' = \infty$.

In this way, we reduced the assumption to the already studied case of a finite p-group. Applying it to $f : S(V'') \to W$, we have

$$\dim Z_f \geq \dim Z_{f_{|S(V'')}} = \infty,$$

which completes the proof of Theorem 4.2.

□

Remark 4.6. One can easily note that our proof of Theorem 4.2 a) is more complicated than the corresponding Borsuk-Ulam theorem presented in [4] and [3]. It is caused by the fact that we need the assumption $S(V)^{P^s} = \emptyset$, which is not necessary in the study of the Borsuk-Ulam problem. Indeed, if $S(V)^{P^s} \neq \emptyset$, then there is no P^s-map from $S(V)$ into $S(W)$, because $S(W)^{P^s} = \emptyset$. The mentioned assumption is necessary to know that $Z_f^{P^s} = \emptyset$, which is necessary to apply Lemma 4.5.

Remark 4.7. Note that the torus $G = \mathbb{T}^k$ is toral, with $p = 1$. Then, the above argument can be applied to any prime p. In this way, we obtain another proof of Theorem 3.13. On the other hand, our proof of Theorem 3.13 is completely elementary. Contrary to it, the proof of Theorem 4.2 uses Proposition 4.3. The latter is the result of [4] and is based on a deep topological result namely the Segal conjecture, proved by G. Carlsson.

Corollary 4.8. *The statement of Theorem 4.2 holds, if we replace* $\dim Z_f$ *by* $\dim Z_f/G$ *in its statement.*

Proof. The statement of corollary follows directly from the main result of [7], since for a compact Lie group G and a metric G-space X, we have dim $X -$ dim $X/G \leq$ dim G. □

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